JOURNAL OF COMBINATORIAL THEORY, Series A 26, 38–47 (1979)

On Tight Spherical Designs

EIICHI BANNAI

Department of Mathematics, The Ohio State University, Columbus, Ohio 43210

Communicated by R. M. Wilson

Received March 21, 1977

According to Delsarte, Goethals, and Seidel, a nonempty subset X of the set of unit vectors in the real vector space $\mathbb{R}^d$ of dimension d is called a *tight spherical t-design* if (i) $\sum_{\alpha \in X} W(\alpha) = 0$ for all homogeneous harmonic polynomials $W(\alpha)$ in $\mathbb{R}^d$ of degree $1, 2, \ldots, t$, and (ii) $|X| = \binom{d+e-1}{d-1} + \binom{d+e-2}{d-1}$, $|X| = 2\binom{d+e-1}{d-1}$ for $t = 2e$ and $t = 2e + 1$, respectively. In this paper, we show that if $t = 2e$, $e \geqslant 3$ or $t = 2e + 1$, $e \geqslant 4$, and if there exists a tight spherical t-design, then the dimension d is bounded by a certain function depending only on t.

Let $\mathbb{R}^d$ be the Euclidean space of dimension d, and let Ω_d be the set of unit vectors in $\mathbb{R}^d$. A nonempty subset X of Ω_d is said to be a spherical t-design (in Ω_d) if $\sum_{\alpha \in X} W(\alpha) = 0$ for all homogeneous harmonic polynomials $W(\alpha)$ in $\mathbb{R}^d$ of degree $1, 2, \ldots, t$ or, equivalently, if the kth moments of X are constant with respect to orthogonal transformations of $\mathbb{R}^d$ for $k = 0, 1, \ldots, t$. (See Delsarte, Goethals, and Seidel [4].) The cardinality of a spherical t-design X is bounded from below, and Delsarte *et al.* [4] have proved that

$$|X| \geqslant \binom{d+e-1}{d-1} + \binom{d+e-2}{d-1}, \qquad |X| \geqslant 2\binom{d+e-1}{d-1},$$

for $t = 2e$ and $t = 2e + 1$, respectively. The spherical t-design is called *tight* if any one of these bounds is attained.

The existence problem of tight spherical designs has been studied by Delsarte, *et al.* [4]. In particular, they constructed some examples of tight spherical t-designs (in Ω_d with $d \geqslant 3$) with $t = 2, 3, 4, 5, 7$, and 11, and they proved the nonexistence of tight spherical 6-designs (in Ω_d, $d \geqslant 3$) by establishing a Lloyd-type theorem. (Note that the tight spherical t-designs in Ω_2 are the regular $(t + 1)$-gons.)

In the present paper we prove the following:

THEOREM A. *Let $t = 2e$, $e \geqslant 3$, or $t = 2e + 1$, $e \geqslant 4$. If there exists a tight spherical t-design in Ω_d, then $d \leqslant d(t)$, i.e., d is bounded by a certain function $d(t)$ depending only on t.*

0097-3165/79/010038-10$02.00/0

This result seems to support the following conjecture slightly stronger than that of Delsarte *et al.* [4]:

Conjecture. There exists no tight spherical t-designs in Ω_d, if $d \geqslant 3$ and $t \neq 1, 2, 3, 4, 5, 7, 11$.

Our proof of Theorem A is done by utilizing the Lloyd-type theorem established by Delsarte *et al.* [4]. Namely, we will analyze the location of the zeros of the (Lloyd-type) polynomials (i.e., certain Jacobi polynomials and certain Gegenbauer polynomials $C_e(x)$ to be explained below), by using the theory of Hermite polynomials. The reader may notice that the present approach is very similar to that of the author's previous paper [2].

1. The Polynomials $R_e(x)$ and $C_e(x)$

For a fixed integer $d \geqslant 3$, we define the polynomials $Q_k(x)$, $R_k(x)$, and $C_k(x)$ ($k = 0, 1, 2,\ldots$) as follows:

$$Q_k(x) := (d + 2k - 2)/(d - 2)\ C_k^{(d-2)/2}(x),$$

$$R_k(x) := \sum_{i=0}^{k} Q_i(x) = (\text{constant}) \cdot P_k^{(\mu+1,\mu)}(x), \qquad \mu = \tfrac{1}{2}(d-3),$$

$$C_k(x) := \sum_{i=0}^{[\frac{1}{2}k]} Q_{k-2i}(x) = C_k^{\frac{1}{2}d}(x).$$

Here, $P_n^{(\alpha,\beta)}$ is the usual Jacobi polynomial, i.e.,

$$P_n^{(\alpha,\beta)}(x) := F(-n, \alpha + n, \beta; x),$$

and $C_n^{\nu}(x)$ is the usual Gegenbauer polynomial, i.e.,

$$C_n^{\nu}(x) := \frac{\Gamma(n + 2\nu)}{n!\ \Gamma(2\nu)}\ P_n^{(2\nu,\nu+\frac{1}{2})}\left(\frac{1-x}{2}\right),$$

where $F(\ ,\ ;x)$ is the Gauss' hypergeometric function and $\Gamma(x)$ is the gamma function. (For more details, see Szegö [7].) It is known that

$$C_{2n}^{\nu}(x) = \frac{(-1)^n\ \Gamma(n+\nu)}{n!\ \Gamma(\nu)}\ F(-n, n+\nu, \tfrac{1}{2}; x^2),$$

and

$$C_{2n+1}^{\nu}(x) = \frac{(-1)^n\ \Gamma(n+\nu+1)}{n!\ \Gamma(\nu)}\ 2x \cdot F(-n, n+\nu+1, \tfrac{3}{2}; x^2),$$

(cf. [7]).

Note that

$$Q_k(1) = \binom{d+k-1}{d-1} - \binom{d+k-3}{d-1},$$

$$R_k(1) = \binom{d+k-1}{d-1} + \binom{d+k-2}{d-1},$$

$$C_k(1) = \binom{d+k-1}{d-1}, \qquad \text{for} \quad k \geqslant 1.$$

Furthermore, note that $Q_k(1)$ are all distinct for $k = 0, 1, 2, \ldots,$ because $Q_{k+1}(1) > Q_k(1)$.

Our whole argument in the present paper is based on the following Lloyd-type theorem for spherical tight t-designs:

THEOREM 1 (Delsarte, Goethals, and Seidel [4]). *Suppose there exists a spherical tight t-design in Ω_d with $d \geqslant 3$. If $t = 2e$, then all the e zeros $x_1, x_2, \ldots, x_e$ of the polynomial $R_e(x)$ must be rational numbers. If $t = 2e + 1$, then all the e zeros $x_1, x_2, \ldots, x_e$ of the polynomial $C_e(x)$ must be rational numbers.*

Remark. Theorem 1 is proved in [4] somewhat implicitly. For an explicit proof of Theorem 1, see [4, Theorem 7.7] and the related parts in [4, 3]. In particular, see [4, Remark 7.6] and [3, in Chap. 2]. Also note that $Q_k(1)$ are all distinct for $k = 0, 1, 2, \ldots.$ The clue of the proof is that all the eigenvalues (i.e., the $P_k(i)$'s in the notation of Delsarte [3]) of the intersection matrices of the association scheme (derived from the spherical tight design) must be rational integers, hence all the numbers $Q_k(i)$ (in the notation of Delsarte [3]) must be rational numbers.

2. PROOF OF THEOREM A FOR THE CASE $t = 2e + 1$, $e \geqslant 4$

Our proof of Theorem A for the case $t = 2e + 1$, $e \geqslant 4$ depends heavily on the following deep result on diophantine equations due to Thue and Siegel (or on the stronger result due to Baker [1]).

SIEGEL'S THEOREM (Siegel [6]). *Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$, a_i are integers, $a_0 \neq 0$, and $n \geqslant 3$. If $f(x)$ has at least three simple zeros (in the complex number field), then the diophantine equation $y^2 = f(x)$ has only finitely many integral solutions in x and y.*

BAKER'S THEOREM (Baker [1]). *Let $f(x)$ be as above. If $f(x)$ has at least three simple zeros, then the absolute values of integral solutions x and y of $y^2 = f(x)$ are bounded effectively by an (explicit) function depending only on the coefficients a_i ($i = 0, 1, \ldots, n$).*

To begin with, we prove the following:

PROPOSITION 2. *If all the zeros $x_1, x_2, \ldots, x_e$ of $C_e^{\frac{1}{2}d}(x)$ are rational numbers, then the following diophantine equation* (1) *or* (2) *must have integral solutions in y (and d):*

$$y^2 = 1 \cdot 3 \cdot 5 \cdots (2n-1)(d+2n)(d+2n+2) \cdots (d+2n+2(n-1)), \quad \text{if} \quad e = 2n. \tag{1}$$

$$y^2 = 3 \cdot 5 \cdots (2n+1)(d+2n+2)(d+2n+4) \cdots (d+2n+2+2(n-1)), \quad \text{if} \quad e = 2n+1. \tag{2}$$

Proof of Proposition 2. Suppose that all the zeros $x_1, x_2, \ldots, x_e$ of $C_e^{\frac{1}{2}d}(x)$ are rational numbers.

First suppose that $e = 2n$ is even. Since $C_{2n}^{\nu}(x) = (\text{const}) \cdot F(-n, n+\nu, \frac{1}{2}; x^2)$, all the $2n$ zeros $x_1, x_2, \ldots, x_{2n}$ $(= x_e)$ of the polynomial $F(x) := F(-n, n+d/2, \frac{1}{2}; x^2)$ must be rational numbers. Note that

$$\begin{aligned} F(x) = 1 &- \binom{n}{1}(d+2n)\,x^2 + \frac{1}{1 \cdot 3}\binom{n}{2} \cdot (d+2n)(d+2n+2)\,x^4 \\ &- \frac{1}{1 \cdot 3 \cdot 5}\binom{n}{3}(d+2n)(d+2n+2)(d+2n+4)\,x^6 + \cdots \\ &\cdots (-1)^n \frac{1}{1 \cdot 3 \cdots (2n-1)}\binom{n}{n}(d+2n)(d+2n+2) \cdots \\ &\cdots (d+2n+2(n-1))\,x^{2n}. \end{aligned}$$

Suppose that $x = a/b$ is a zero of $F(x)$, where a and b are relatively prime integers. Then a^2 must be a divisor of $1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)$, because if we multiply $F(x) = 0$ by $1 \cdot 3 \cdot \cdots \cdot (2n-1) \cdot b^{2n}$, then all the terms except for the first term $1 \cdot 3 \cdot 5 \cdot \cdots (2n-1) \cdot b^{2n}$ are divisible by a^2. Therefore, there exists a positive integer A such that $A^2 \mid 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)$ and $x_i = A/m_i$ where m_i are integers $(i = 1, 2, \ldots, 2n)$. Now, by looking at the equation $F(x) = 0$, we get that

$$\prod_{i=1}^{2n} \frac{1}{x_i} = \prod_{i=1}^{2n} \frac{m_i}{A} = \frac{(d+2n)(d+2n+2) \cdots (d+2n+2(n-1))}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}.$$

On the other hand, since $F(x)$ is an even function, $F(x_i) = 0$ implies $F(-x_i) = 0$. Therefore,

$$\prod_{i=1}^{2n} \frac{m_i}{A} = \left(\frac{\prod_{i=1}^{n} m_i}{A^n}\right)^2 = \frac{(d+2n)(d+2n+2) \cdots (d+2n+2(n-1))}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}.$$

Here we assume without loss of generality that $x_1, \dots, x_n$ are positive, and $x_{n+1}, \dots, x_{2n}$ are negative. Now if we put

$$y = \frac{1}{A^n} \cdot \prod_{i=1}^{n} m_i \cdot 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1),$$

then

$$y^2 = 1 \cdot 3 \cdot 5 \cdots (2n-1)(d+2n)(d+2n+2) \cdots (d+2n+2(n-1)),$$

and y must be an integer. Thus we have proved Proposition 2 for all even e.

Next, suppose that $e = 2n+1$ is odd. Since $C_{2n+1}^{\nu}(x) = (\text{const}) \cdot x \cdot F(-n, n+\nu+1, \frac{3}{2}; x^2)$, we obtain that there is a positive integer A such that $A^2 \mid 3 \cdot 5 \cdot \cdots \cdot (2n+1)$ and the $2n$ zeros $x_1, \dots, x_{2n}$ of $F(x) := F(-n, n+d/2+1, \frac{3}{2}; x^2)$ are equal to $x_i = A/m_i$, where m_i are integers ($i = 1, 2, \dots, 2n$). Similarly, we get

$$\prod_{i=1}^{2n} \frac{m_i}{A} = \frac{(d+2n+2)(d+2n+4) \cdots (d+2n+2+2(n-1))}{3 \cdot 5 \cdot \cdots \cdot (2n+1)}.$$

Similarly, if we put $y = \prod_{i=1}^{n} m_i/A$ (where m_i's are positive), we get $y^2 = 3 \cdot 5 \cdot \cdots \cdot (2n+1)(d+2n+2)(d+2n+4) \cdots (d+2n+2+2(n-1))$, and y must be an integer. This completes the proof of Proposition 2.

Proof of Theorem A for the Case $t = 2e+1$, $e \geqslant 4$

First, we assume that $e \geqslant 6$. Then, by Theorem 1, all the zeros $x_1, x_2, \dots, x_e$ of $C_e(x)$ $(= C_e^{\frac{1}{2}d}(x))$ must be rational numbers. By Proposition 2, the diophantine equation (1) or (2) (in Proposition 2) must have integral solutions in y and d. If we put $f(d)$ equal to the right-hand side of Eq. (1) or (2), then $f(d)$ is a $[e/2]$th degree polynomial in d ($[e/2] \geqslant 3$) and clearly satisfies the assumption of the Theorem of Siegel and the Theorem of Baker. Therefore, d is bounded by a certain function depending only on e (i.e., depending only on t). This proves Theorem A for $e \geqslant 6$.

Next, suppose that $e = 5$. Then the number A (defined in the proof of Proposition 2) is equal to 1. Therefore, by Theorem 1, all the solutions in m of the equation

$$m^4 - \frac{2}{3}(d+6)\, m^2 + \frac{(d+6)(d+8)}{3 \cdot 5} = 0$$

must be integers. Note that the above equation is obtained from $x = 1/m$ and $F(x)$ $(:= F(-2, 3+d/2, \frac{3}{2}; x^2)) = 0$. If we put $d+6 = d_0$, then

$$15m^4 - 10d_0m^2 + d_0(d_0+2) = 0,$$

i.e.,

$$d_0^2 - 2(5m^2-1)\, d_0 + 15m^4 = 0.$$

Since d_0 must be an integer,

$$(5m^2 - 1)^2 - 15m^4 = 10m^4 - 10m^2 + 1$$

must be a square of an integer. Again, applying the Theorem of Siegel (or the Theorem of Baker), we get that m is bounded by a certain constant number. This, in turn, implies that d must be bounded by a certain constant number. Thus we have proved Theorem A for $e = 5$ (and $t = 11$). (Note that from $d = 24$, we actually get an example of spherical tight 11-designs (cf. [4]).)

Suppose that $e = 4$. Then, again $A = 1$, and all the solutions in m of the equation

$$m^4 - 2(d + 4)m^2 + (d + 4)(d + 6)/3 = 0$$

must be integers. Putting $d + 4 = d_0$, we get

$$3m^4 - 6d_0m^2 + d_0(d_0 + 2) = 0.$$

Hence,

$$(3m^2 - 1)^2 - 3m^4 = 6m^4 - 6m^2 + 1$$

must be a square of an integer. Then, by the Theorem of Siegel (or the Theorem of Baker), m, and consequently d, must be bounded by a certain constant number. This completes the proof of Theorem A for $t = 2e + 1$, $e \geqslant 4$.

Remark. If one can show the impossibility of nontrivial integral solutions of Eqs. (1) and (2) (in Proposition 2), for all $e \geqslant 6$, then it will imply the nonexistence of tight spherical $(2e + 1)$-designs ($e \geqslant 6$) in Ω_d ($d \geqslant 3$). This diophantine equation problem itself seems to be interesting.

3. Proof of Theorem A for the Case $t = 2e, e \geqslant 3$

Our proof of Theorem A for $t = 2e$, $e \geqslant 3$ differs very much from the proof of Theorem A for $t = 2e + 1$, $e \geqslant 4$, given in the previous section. Our approach in the present section is similar to that in the author's previous paper [2].

Because of Theorem 1, we have only to prove the following:

PROPOSITION 3. Let $t = 2e$, $e \geqslant 3$. If all the zeros $x_1, x_2, \ldots, x_e$ of the polynomial $R_e(x)$ are rational numbers, then d is bounded by a certain function depending only on e.

Before starting the proof of Proposition 3, we mention the following result due to Schur.

LEMMA 4. (Schur [5]). *Let*

$$H_m(x) := \sum_{\mu=0}^{[m/2]} (-1)^\mu \binom{m}{2\mu} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2\mu - 1)\, x^{m-2\mu}$$

be the Hermite polynomial of degree m. Then the polynomials

$$x^{-1}H_3(x),\ H_4(x),\ x^{-1}H_5(x),\ H_6(x), \ldots,$$

are all irreducible over the rational number field Q.

COROLLARY 5. *Let $m \geqslant 4$. Let ξ_1 and ξ_2 ($\xi_i \neq 0$, $\xi_1 \neq \pm\xi_2$) be two zeros of $H_m(x)$. Then $1/\xi_1^2 - 1/\xi_2^2$ is not a rational number.*

Corollary 5 is easily derived from Lemma 4.

Proof of Proposition 3. To begin, note that $R_e(x) = C_e(x) + C_{e-1}(x) = C_e^{\frac{1}{2}d}(x) + C_{e-1}^{\frac{1}{2}d}(x)$.

First suppose that $e = 2n$ is even. Then

$$\begin{aligned} R_e(x) &= C_e^{\frac{1}{2}d}(x) + C_{e-1}^{\frac{1}{2}d}(x) \\ &= (\text{const}) \cdot \{F(-n, n + d/2, \tfrac{1}{2}\,; x^2) \\ &\quad - 2nx \cdot F(-(n-1), n + d/2, \tfrac{3}{2}\,; x^2)\}. \end{aligned}$$

Let us set

$$\psi(x) = F(-n, n + d/2, \tfrac{1}{2}; x^2) - 2nx \cdot F(-(n-1), n + d/2, \tfrac{3}{2}; x^2).$$

Then $\psi(x)$ is a polynomial of degree $2n$ in x, and of degree n in d. Let $x = 1/z$. Then $\bar{\psi}(z) := z^{2n}\, \psi(z)$ is a polynomial of degree $2n$ in z and of degree n in d. Clearly x is a zero of $\psi(x)$ if and only if z is a zero of $\bar{\psi}(z)$. We can write

$$\bar{\psi}(z) = \sum_{i,j \geqslant 0} a_{ij} z^i d^j, \qquad \text{where} \quad a_{ij} \in Q.$$

Clearly, $a_{ij} = 0$ if $i + 2j > 2n$. Let

$$\bar{\psi}(z) = \bar{\psi}_{2n}(z) + \bar{\psi}_{2n-1}(z) + \cdots + \bar{\psi}_0(z),$$

where

$$\bar{\psi}_k(z) := \sum_{2i+j=k} a_{ij} z^i d^j.$$

Now, by noting that

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \frac{x^{2\nu}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2\nu - 1)} = (-1)^n \cdot \frac{H_{2n}(x)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}, \tag{3}$$

and

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \frac{x^{2\nu}}{3\cdot 5\cdot \cdots \cdot (2\nu+1)} = (-1)^n \cdot x^{-1} \cdot \frac{H_{2n+1}(x)}{3\cdot 5\cdot \cdots \cdot (2n+1)}, \tag{4}$$

we easily get that

$$\bar{\psi}_{2n}(z) = (-1)^n \frac{z^{2n} H_{2n}(d^{1/2}/z)}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)},$$

and

$$\bar{\psi}_{2n-1}(z) = (-1)^{n-1} \frac{-2\cdot (1/d^{1/2})\cdot z^{2n} H_{2n-1}(d^{1/2}/z)}{3\cdot 5\cdot \cdots \cdot (2n-1)},$$

where $H_m(x)$ is the Hermite polynomial of degree m defined in Lemma 4. Now, let $\xi_1, \xi_2, \ldots, \xi_{2n}$ be the zeros of $H_{2n}(x)$. Let us introduce new variables ϵ_i by $z_i = d^{1/2}/\xi_i + \epsilon_i$, where z_i $(i = 1, 2, \ldots, 2n)$ are the zeros of $\bar{\psi}(z)$. Now, if d becomes sufficiently large (i.e., if $d \to +\infty$), then ϵ_i approaches λ_i (i.e., $\epsilon_i \to \lambda_i$), where

$$\lambda_i = \frac{\bar{\psi}_{2n-1}(d^{1/2}/\xi_i)}{\bar{\psi}'_{2n}(d^{1/2}/\xi_i)}.$$

(Note that $\bar{\psi}_{2n}(z) := (d/dz)\, \bar{\psi}_{2n}(z)$.) Since $H'_m(x)$ $(:=(d/dx)\, H_m(x)) = m \cdot H_{m-1}(x)$, we have

$$\begin{aligned}\lambda_i &= \frac{-2n\cdot (1/d^{1/2})\cdot (d^{1/2}/\xi_i)^{2n}\cdot H_{2n-1}(\xi_i)}{2n\cdot (d^{1/2}/\xi_i)^{2n}\cdot \dfrac{d^{1/2}}{(d^{1/2}/\xi_i)^2}\, H_{2n-1}(\xi_i)}\\ &= \frac{-1}{\xi_i^2}.\end{aligned}$$

Now, the argument in the proof of Proposition 2 shows that there exists a positive integer A such that $A^2 \mid 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)$ and that any zero z_i of $\bar{\psi}(z)$ is expressed as $z_i(=1/x_i) = m_i/A$, where m_i $(i = 1, 2, \ldots, 2n)$ are integers. Since $\bar{\psi}(z)$ is an even function, we may assume without loss of generality that $z_2 = -z_1$ and $z_4 = -z_3$ $(z_1 \neq z_3)$. Since each z_i is in $\mathbb{Z}/A = \{(i/A) \mid i \in \mathbb{Z}\}$, $z_1 + z_2 - z_3 - z_4$ lies in $\mathbb{Z}/A$. On the other hand, if $d \to +\infty$, then $z_i \to d^{1/2}/\xi_i - 1/\xi_i^2$. Therefore

$$z_1 + z_2 - z_3 - z_4 \to -2(1/\xi_1^2 - 1/\xi_2^2).$$

From Corollary 5, $2(1/\xi_1^2 - 1/\xi_2^2)$ is not a rational number. In particular, $2(1/\xi_1^2 - \xi_2^2)$ is not contained in $\mathbb{Z}/A$ and is of positive distance with the subset $\mathbb{Z}/A$. Therefore, d cannot become too large, i.e., d must be bounded by a certain function depending only on e.

Next, suppose that $e = 2n + 1$ is odd. Then

$$\begin{aligned} R_e(x) &= C_e^{\frac{1}{2}d}(x) + C_{e-1}^{\frac{1}{2}d}(x) \\ &= (\text{const}) \cdot \{(n + d/2 \cdot 2x \cdot F(-n, n + d/2 + 1, \tfrac{3}{2}\,; x^2) \\ &\quad + F(-n, n + d/2, \tfrac{1}{2}\,; x^2)\}. \end{aligned}$$

Let

$$\psi(x) = (n - d/2) \cdot 2x \cdot F(-n, n + d/2 + 1, \tfrac{3}{2}; x^2) + F(-n, n\ d/2, \tfrac{1}{2}; x^2).$$

Let $x = 1/z$. Then $\bar{\psi}(z) := z^{2n+1}\,\psi(z)$ is a polynomial of degree $2n + 1$ in z, and x is a zero of $\psi(x)$ if and only if z is a zero of $\bar{\psi}(z)$. Let $\bar{\psi}(z) = \sum_{i,j\geqslant 0} a_{ij}z^i d^j$, where $a_{ij} \in \mathbb{Q}$. Clearly $a_{ij} = 0$ if $i + 2j > 2n + 1$. Let $\bar{\psi}(z) = \bar{\psi}_{2n+1}(z) + \bar{\psi}_{2n}(z) + \cdots + \bar{\psi}_0(z)$, where $\bar{\psi}_k(z) := \sum_{i+2j=k} a_{ij}z^i d^j$. Now, we get (cf. formulas (3) and (4) above)

$$\bar{\psi}_{2n+1}(z) = (-1)^n \cdot \frac{d^{1/2} \cdot z^{2n+1} \cdot H_{2n+1}(d^{1/2}/z)}{3 \cdot 5 \cdot \cdots \cdot (2n + 1)},$$

and

$$\bar{\psi}_{2n}(z) = (-1)^n \cdot \frac{z^{2n+1} \cdot H_{2n}(d^{1/2}/z)}{1 \cdot 3 \cdot \cdots \cdot (2n - 1)}.$$

Let $\xi_1, \xi_2, \ldots, \xi_m$ be the nonzero zeros of $H_{2n+1}(x)$. Then, if $d \to +\infty$, then $2n$ zeros $z_1, z_2, \ldots, z_{2n}$ (among $2n + 1$ zeros of $\bar{\psi}(z)$) approach $d^{1/2}/\xi_i + \lambda_i$, where

$$\begin{aligned} \lambda_i &= \frac{\bar{\psi}_{2n}(d^{1/2}/\xi_i)}{\bar{\psi}'_{2n+1}(d^{1/2}/\xi_i)} \\ &= \frac{(d^{1/2}/\xi_i)^{2n+1} \cdot H_{2n}(\xi_i) \cdot (2n + 1)}{-2n \cdot (d^{1/2}/\xi_i)^{2n+1} \cdot \dfrac{(d^{1/2})^2}{(d^{1/2}/\xi_i)^2} \cdot H_{2n}(\xi_i)} \\ &= \frac{2n + 1}{2n} \cdot \frac{1}{\xi_i^{\,2}}. \end{aligned}$$

Now, we may assume that $e = 2n + 1 \geqslant 5$ because the case $e = 3$ is already settled completely in [4, Theorem 7.7]. Now, the same argument as before shows that $z_1 + z_2 - z_3 - z_4$ is in $\mathbb{Z}/A$, where $A^2 \mid 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)$. But if $d \to +\infty$, then $z_1 + z_2 - z_3 - z_4 \to -((2n + 1)/n)(1/\xi_1^{\,2} - 1/\xi_2^{\,2})$, and the limit is not a rational number. However, this is impossible, hence d is bounded by a certain function depending only on e.

Thus, Theorem A has been completely proved.

ACKNOWLEDGMENT

The author thanks Ph. Delsarte for sending him the preprint of [4] before publication.

Added in proof. While this paper was waiting to be published, improved results were obtained jointly by R. M. Damerell and the present author. As for the details, see the forthcomming two papers: Tight spherical designs, I (to appear in *J. Math. Soc. Japan*), and II (submitted for publication) by E. Bannai and R. M. Damerell.

References

1. A. Baker, Bounds for the solution of the hyperelliptic equation, *Proc. Cambridge Philos. Soc.* **65** (1969), 439–444.
2. E. Bannai, On perfect codes in the Hamming schemes $H(n, q)$, with q arbitrary. *J. of Combinatorial Theory A* **23** (1977), 52–67.
3. P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Res. Repts.* Suppl. No. 10 (1973).
4. P. Delsarte, J. M. Goethals, and J. J. Seidel, Spherical codes and designs, *Geometriae Dedicata,* **6** (1977), 363–388.
5. I. Schur, Affektolose Gleichungen in der Theorie der Laguerreschen und Hermiteschen Polynome, *J. Reine Angew. Math.* **165** (1931), 52–58.
6. C. Siegel (under pseudonym X), The integral solution of the equation $y^2 = ax^n + b^{n-1} + \cdots + k$, *J. London Math. Soc.* **1** (1920), 66–68.
7. G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc. Colloq. Publ., Vol. 23, 1st ed. (1939); 4th ed. (1975).